Notes for the lecture ”singular spaces and the Poincaré-duality“ by Markus Banagl

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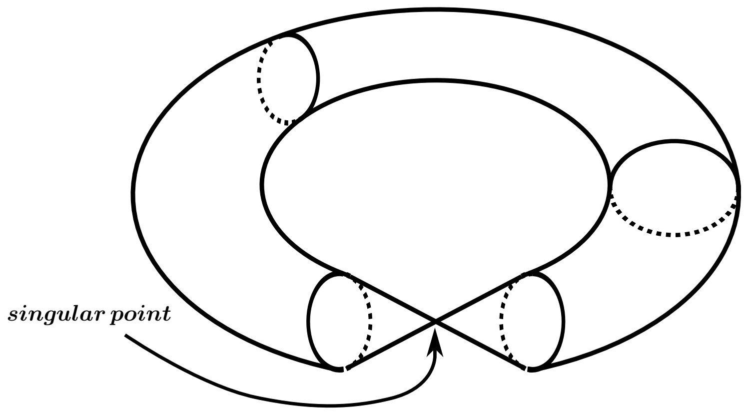
Figure 1: Pinched Torus



1 Motivation

Theorem 1.1 (Poincaré-duality for manifolds)**.** *Let M^n be a closed, oriented manifold.*

$$[M] \in H_n(M)$$

$$- \cap [M] \colon H^i(M) \xrightarrow{\cong} H_{n-i}(M)$$

In particular

$$b_i(M) := \mathrm{rnk} H_i(M) = \mathrm{rnk} H_{n-i}(M)$$

Singular spaces do not posses Poincaré-duality.
Example: "Pinched Torus" (Fig. 1)

$$H_0(X) \cong \mathbb{Z}$$

$$H_2(X) \cong \mathbb{Z}$$

Problem: $H_1(X) = \mathbb{Z}$. There is no intesersection product, so Poincaré-duality does not hold.
Example: $X^3 = \Sigma(T^2)$, the suspension of the Torus (Fig. 2).

$$b_0 = 1 = b_3$$

$$b_1 = 0$$

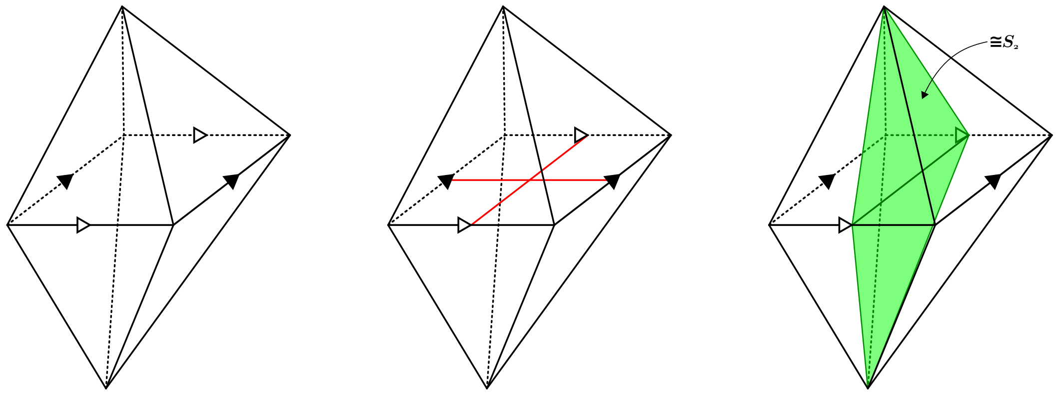
$$b_2 = 2$$

Since $b_1 \neq b_2$, Poincaré-duality does not hold.

2 Motivational examples of stratified spaces

A **stratification** of a singular space X will turn out to be a Decomposition of X into a disjoint union of S_i, such that the S_i are manifolds and points in S_i are "equaly singular".

Figure 2: Suspended Torus



(a) Suspension of the (flat) Torus

(b) The meridians of the torus are 1-cells

(c) We can suspend the meridians and get 2-cells

2.1 Topological construction

- A Cross (Fig. 4a)
- The pinched torus (Fig. 4b)
- $X^3 = \Sigma T^2 = \{N, S\} \cup \underbrace{\left(X^3 \backslash \{N, S\}\right)}_{\cong (0,1) \times T^2}$
- $X^2 = T^2 \cup T^2 = S_1 \cup S_2$ (Fig. 4c)
- $X^4 = S^1 \times \Sigma T^2$
- $X^4 = \Sigma\Sigma T^2$ (Fig. 4d)

2.2 Smooth groupactions

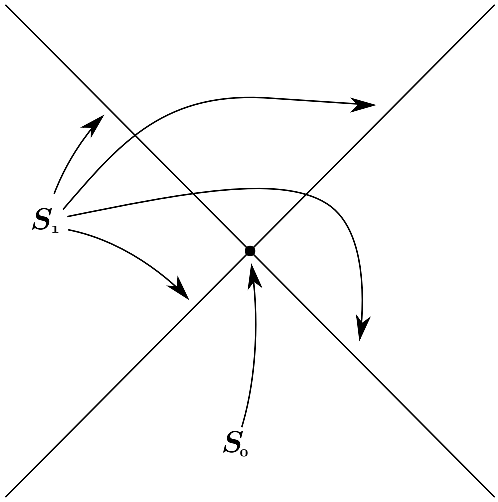
Let M^n be a closed smooth manifold and G a compact Lie group acting smoothly on M.
Example: $G = S^1, M = S^2$
$\rightsquigarrow$ Orbit Space M/G
The orbit space will usually be singular.
We will stratify M/G. Let H be a closed Subgroup $H < G$.

$$M_{(H)} := \{x \in M \mid G_x \sim H\}$$

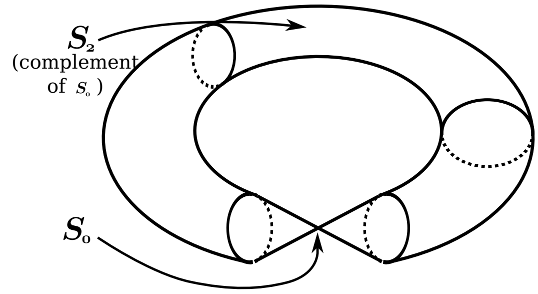
Where $\sim$ is conjugate equivalence, G_x is the isotropy group at x and (H) is the conjugate class of H.
$M_{(H)}$ is a union of orbits:

$$G_{g \cdot x} = gG_xGg^{-1} \sim G_x \sim H$$
$$\Rightarrow G_{g \cdot x} \sim H$$

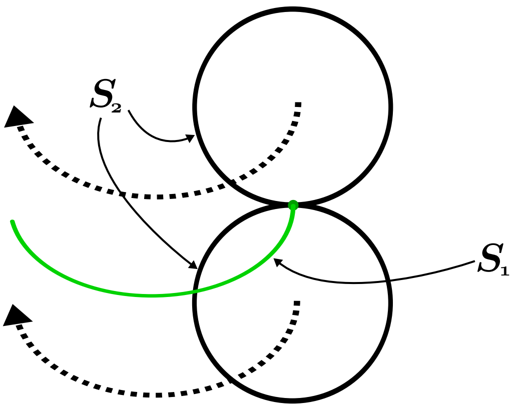
Figure 3: Topological constructions



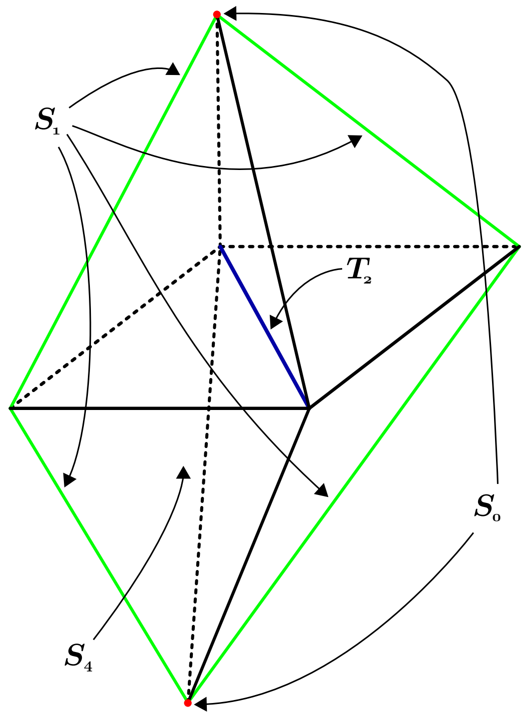
(a) Cross



(b) Stratification of the pinched torus



(c) Connected tori



(d) Double-suspension of the torus

Proposition 2.1. *Assume M/G is connected. Then there exists a unique conjugate class (H_0), such that $M_{(H_0)}$ is open and dense in M.*

Example: In the rotating sphere example, $(H_0) = (e_0)$, because $M_{(e)} = S^2 \backslash \{N, S\}$.

- $M_{(H)}$ is a smoothly embedded submanifold of M, called the orbit bundle of (H).

$$\begin{array}{ccc} G/H & \longrightarrow & M_{(H)} \\ & & \Big\downarrow \pi \\ & & M_{(H)}/G \end{array}$$

- As G is compact, there are only finitely many isotropy types.

$$M = \bigcup_{i=1}^{n} M_{(H_i)}$$

$$M/G = \bigcup_{i=1}^{n} \underbrace{M_{(H_i)}/G}_{=S_{(H)}}$$

Example: In the rotating sphere example, $M/G = M_{(e)}/G \cup \{N, S\}/G$.

Link

Suppose $(H_0) = (e)$ (i.e. the area, where G acts freely is open and dense), there are only 2 isotropy types, (e) and (H), $H \neq e$, and G acts transitively on $M_{(H)}$ (so there is some $x \in M_{(H)}$, such that $M_{(H)} = Gx$).

Question: What is the link of the singular point in M/G.

$$V_x := \frac{T_x M}{T_x(Gx)}$$

V_x is called a "slice".
Let $h \in G_x \sim H$.

$$h_*\colon T_x(Gx) \to T_{hx}(Gx) = T_x(Gx)$$

We obtain a represantation

$$H \sim G_x \to \mathrm{GL}(V_x)$$

called the **slice-representation**.
We can therefore form

$$G \times_{G_x} V_x := (G \times V_x)/G_x$$

which is a vector bundle over Gx.

Theorem 2.2 (Slice theorem)**.** *There exists an open neighbourhood N_x of the orbit Gx and an equivariant diffeomorphism*

$$N_x \cong G \times_{G_x} V_x$$

Equivariant means, that the diffeomorphism preserves the group-action.
Since G_x is compact, there exists a G_x-invariant riemannian metric on V_x which induces a metric on $G \times_{G_x} V_x =: E$.
We then define the unit sphere bundle as

$$SE := \{v \in E \mid \|v\| = 1\}$$

$$\begin{array}{ccc} S_1 & \longrightarrow & E \\ & & \downarrow \\ & & Gx \end{array}$$

Then the link of the singular point in M/G is SE/G.
If G does not act transitively on $M_{(H)}$, then

$$\text{link} = S\nu_{M_{(H)}}|_{Gx}/G$$

2.3 Triangulated Spaces (simplicial complexes)

Let

$$X^n = S_0 \cup S_1 \cup ... \cup S_n$$

be a triangulated space.
Let S_i be the union of interiors of i-simplices in X. Then S_i is an i-dimenionsal manifold.
Question: What is the link?
Let T be simplicial complex, such that $|T| = X$. Let T' be the first barycentric subdivision of T. Then every simplex Δ in T has a "dual block" $D(\Delta)$, which is a union of simplices in T'.

$$\text{link}(\Delta) = \partial D(\Delta)$$

2.4 Algebraic varieties (over $\mathbb{R}$)

"Singularities of algebraic varieties remain almost completely misterious" (J. Harris)
Reason: Among the singularities of affine varieties $\subset \mathbb{A}^n$ are the cones on projective varieties $Y \subset \mathbb{P}^{n-1}$. So to have a complete understanding of affine singularities, one would need first a complete classification of projective varieties.
Suppose we have a projective variety Y.

$$\Theta : \mathbb{A}^n \backslash 0 \to \mathbb{P}^{n-1}$$

$$(x_1, ..., x_n) \mapsto [x_1, ..., x_n]$$

Then

$$\text{cone}(Y) = \Theta^{-1}(Y) \cup \{0\}$$

is an affine variety, whose ideal is the ideal $I(Y)$.

Whitney's Algorithm

(Elementary structure of real algebraic varieties)

1. Let I be an ideal in $\mathbb{R}[x_1, ..., x_n]$, $p \in \mathbb{R}^n$.

$$A = \{(\nabla f)(p) \mid f \in I\} \subset \mathbb{R}^n$$

is a real vectorspace.

$$\text{rnk}_p(I) := \dim A$$

2. Let $S \subset \mathbb{R}^n$ be a subset.

$$I(S) = \{f \in \mathbb{R}[X] \mid f|_S \equiv 0\}$$
$$\text{rnk}_p S := \text{rnk}_p I(S)$$
$$\text{rnk} S := \max_{p \in S} \text{rnk}_p S$$

3. Let V be a real algebraic variety, $V \subset \mathbb{R}^n$. Set

$$M_1 := \{p \in V \mid \text{rnk}_p V = \text{rnk} V\}$$

Then M_1 is an agrebraic manifold of dimension

$$\dim M_1 = n - \text{rnk} V$$

Set $V_1 := V \backslash M_1$. Then either $V_1 = \varnothing$, or V_1 is an algebraic subvariety of V.
You can then proceed recursively using V_1 instead of V and obtain a decomposition

$$V = M_1 \cup M_1 \cup ... \cup M_k$$

in algebraic manifolds.

Example: Let $V = \{x^2 - y^2 z = 0\} \subset \mathbb{R}^n$ (the "Whitney umbrella").

$$I(V) = (x^2 - y^2 z)$$

$$A = \left\{ \begin{pmatrix} 2x \\ -2yz \\ -y^2 \end{pmatrix} \right\} = \begin{cases} 0 & x = y = 0 \\ 1 & x \neq 0 \vee y \neq 0 \end{cases}$$

$$\text{rnk}_p V = \begin{cases} 0 & p \in z-\text{axis} \\ 1 & p \notin z-\text{axis} \end{cases}$$

$$\text{rnk} V = 1$$

$$\Rightarrow M_1 = V \backslash (z-\text{axis}), \dim M_1 = 3 - 1 = 2$$

$$V_1 = V \backslash M_1 = (z-\text{axis})$$

$$I(V_1) = (x,y), A = \left\langle \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix} \right\rangle$$

$$\text{rnk} V_1 = 2 = \text{rnk}_p V_1, \forall p \in V_1$$

$$\Rightarrow M_2 = V_1 = (z - \text{axis}), \dim M_2 = 3 - 2 = 1$$

$$\Rightarrow V = M_1 \cup M_2 = (V \backslash (z - \text{axis})) \cup (z - \text{axis})$$

Note: $M_1 \subset \{z \geqslant 0\}$.
Remark: Most examples so far satisfied the "frontier condition":
If $S_i \cap \overline{S_j} \neq \emptyset$, then $S_i \subset S_j$.
Problem: The frontier condition fails for the given decomposition of V.
We can fix this, by refining the stratification:

$$V = (V - (z - \text{axis})) \cup \overline{\{z > 0\} \cup \{z = 0\} \cup \{z < 0\}}^{x=y=0}$$

3 Stratified spaces

Convention: $c(X)$ will be the closed cone on a topological space X, namely

$$c(X) = [0,1] \times X/(\{0\} \times X)$$

$c^\circ(X)$ will be the open cone, namely

$$c^\circ(X) = [0,1) \times X/(\{0\} \times X)$$

$$c(\emptyset) = p$$

Definition 3.1 (Topoloical stratified pseudomanifold)**.** *A 0-dimensional topological stratified pseudomanifold is a countable set of points with the discrete topology.*
An n-dimensional $(n > 0)$ ***topological stratified pseudomanifold*** *is a paracompact, Hausdorffspace* X^n *together with a filtration by closed subsets*

$$X = X_n \supset X_{n-1} = X_{n-2} \supset X_{n-3} \supset ... \supset X_0 \supset X_{-1} = \emptyset,$$

such that

1. *All nonempty differences* $X_i \backslash X_{i-1}$ *are topological manifolds of dimension i.*

2. $X \backslash X_{n-2}$ *(the "top stratum") is (open and) dense in X.*

3. ***Local triviality:*** $\forall k$*: Let* $x \in X_{n-k} \backslash X_{n-k-1}$*. There exists an open neighbourhood* U_x *of x in X and a compact,* $(k-1) < n$ *dimensional topological stratified pseudomanifold*

$$L = L_{k-1} \supset L_{k-3} \supset L_{k-4} \supset ... \supset L_0 \supset L_{-1}$$

and a homeomorphism

$$\phi : U_x \xrightarrow{\cong} \mathbb{R}^{n-k} \times c^\circ(L^{k-1})$$

which is ***stratum-preserving****.*

L is called a ***link*** *at x of the connected component of* $X_{n-k} \backslash X_{n-k-1}$ *that contains x.*

Definition 3.2. *A* ***PL-space*** *(piecewise linear) is a topological space X together with a class $\mathcal{T}$ of locally finite triangulations of X, such that*

1. *$\mathcal{T}$ is closed under linear subdivision.*
2. *any two triangulations $T, T' \in \mathcal{T}$ have a common linear subdivision $T'' \in \mathcal{T}$.*

Definition 3.3. *A closed subset $A \subset X$ is a* ***closed PL-subspace*** *if there is a $T \in \mathcal{T}$, such that A is given by a subcomplex of T.*

Definition 3.4. *A* ***PL-stratified pseudomanifold*** *is...*

- *X_i are closed PL-subspaces.*
- *Use "PL-manifold" instead of (top.) manifold.*
- *Use "PL-homeomorphism" instead of homeomorphism.*

Example: A classical PL-pseudomanifold is a simplicial complex such that every simplex is the face of some n-simplex and every $(n-1)$-simplex is the face of precisely two n-simplices. Any such classical PL-pseudomanifold can be equipped with a stratification making it a PL-stratified pseudomanifold in the sense of the definition.
Example: Any irreducible complex agebraic variety is a stratified pseudomanifold (strata have only even dimension). Similarly for real algebraic varieties.
Let X^n be a PL-stratified pseudomanifold with PL-structure $\mathcal{T}$. For $i = 0, 1, 2, ...$

$$C_i^T(X) := \{\xi : \{\text{ oriented } i\text{-simplices in } T\} \to \mathbb{Z} \mid \xi(-\sigma) = -\xi(\sigma)\}$$

is the abelian group of i-chains. Infinite chains are allowed.
Remark: We allow infinite chains because eventually we need restriction maps

$$C_i(X) \to C_i(U), U \underset{\text{open}}{\subset} X$$

If T' is a subdivision of T, then there is a canonical map

$$C_i^T(X) \to C_i^{T'}(X)$$

Definition 3.5.

$$C_i(X) = \varinjlim_{T \in \mathcal{T}} C_i^T(X)$$

($\xi \in T$ and $\xi' \in T'$ are ***equivalent****, if thee is a common subdivision $T''\mathcal{T}$ such that the image of ξ under $C_i^T \to C_i^{T''}$ equals the image of ξ' under $C_i^{T'} \to C_i^{T''}$)*
$\xi \in C_i(X)$ has a ***support*** *$|\xi|$, which is a closed PL-subspace.*

Definition 3.6 (Borel-Moore-homology)**.**

$$H_i(X) := H_i(C_*(X))$$

Warning: The Borel-Moore-homology is not a homotopy-invariant
Example: $X = \mathbb{R}^1$.

$$H_0(\mathbb{R}) = 0, H_0(\mathbb{R}^1) = \mathbb{Z}$$

Recall from linear algebra: $A, B \subset \mathbb{R}^n$ are **transverse** to each other ($A \pitchfork B$), if

$$\dim(A \cap B) \leqslant \dim A + \dim B - n$$

Theorem 3.7 (McCrory's theorem)**.** *The complex*

$$IC_i^{\bar{0}}(X) := \{\xi \in C_i(X) \mid |\xi| \text{ is transverse to the strata of } X\}$$

$$= \{\xi \in C_i(X) \mid \dim(|\xi| \cap X_{n-k} \leqslant \dim \xi + \dim X_{n-k} - \dim X = i - k\}$$

computes the cohomology $H^{n-i}(X)$.

If it were possible to move every chain ξ into transverse position with the strata, then ordinary (co)homology of X would satisfy the poincaré-duality.
But poincaré-duality fails for singular X, so in general it is not possible to move chains transversely to the strata.
Idea: (Goresky-MacPherson) Introduce a parameter, called a "perversity-function" that measures the extent to which a chain ξ is allowed ot deviate from full transversality.

Definition 3.8 (Perversity)**.** *A **perversity** is a fucntion*

$$\bar{p}\colon \{2, 3, 4, ...\} \to \{0, 1, 2, ...\}$$

such that

1. $\bar{p}(2) = 0$
2. $\bar{p}(k) \leqslant \bar{p}(k+1) \leqslant \bar{p}(k) + 1$

Example: The zero-perversity $\bar{0}(k) = 0$
Example: The top-perversity $\bar{t}(k) = k - 2$
Example: The lower-middle-perversity $\bar{m} = (0, 0, 1, 1, 2, 2, ...)$
Example: The upper-middle-perversity $\bar{n} = (0, 1, 1, 2, 2, 3, ...)$

Definition 3.9. *Two perversities $\bar{p}, \bar{q}$ are called **complementary**, if*

$$\forall k\colon \bar{p}(k) + \bar{q}(k) = k - 2$$

Example: $\bar{0}, \bar{t}$ are complementary.
Example: $\bar{m}, \bar{n}$ are complementary.

Definition 3.10. *Let $\bar{p}$ be a perversity. The **intersection chain complex** is*

$$IC_i^{\bar{p}}(X) := \{\xi \in C_i(X) \mid \dim(|\xi| \cap X_{n-k}) \leqslant i - k + \bar{p}(k) \wedge \dim(|\partial\xi \cap X_{n-k}) \leqslant i - 1 - k - \bar{p}(k)\}$$

$\Rightarrow IC_*^{\bar{p}}(X)$ is a chain complex with the usual boundary maps.

Definition 3.11. *The **intersection homology groups** are:*

$$IH_i^{\bar{p}}(X) := H_i(IC_*^{\bar{p}}(X)) = \ker \partial_i / \operatorname{Im} \partial_{i-1}$$

Write $\bar{p} \leqslant \bar{q}$, if $\bar{p}(k) \leqslant \bar{q}(k), \forall k$.
If $\bar{p} \leqslant \bar{q}$, then $IC_*^{\bar{p}}(X) \subset IC_*^{\bar{q}}(X)$. This inclusion induces a canonical map

$$IH_*^{\bar{p}}(X) \to IH_*^{\bar{q}}(X)$$

Definition 3.12. *X^n is* ***orientable*** *if the n-simplicies in some admissable triangulation of X can be oriented, such that their sum is a cycle.*
$[X] \in H_n(X)$ is called the ***fundamental class*** *of X.*

Proposition 3.13. *Let X^n be an orientad, compact, PL-stratified pseudomanifold. Then*

$$- \cap [X] : H^i(X) \to H_{n-i}(X)$$

factors through $IH^{\bar{0}}_{n-i}(X)$:

$$\begin{array}{ccc} H^i(X) & \xrightarrow{\cap[X]} & H_{n-i}(X) \\ & {\scriptstyle \cap[X]} \searrow \quad \nearrow & \\ & IH^{\bar{0}}_{n-i}(X) & \end{array}$$

Proof. Under $- \cap [X]$, the characteristic cochain of an i-simplex σ in an admissable triangulation T of X is mapped to the dual block $D(\sigma)$ in the first barycentric subdivision T' of T.
But the dualblock of a simplex is indeed transverse. □

We even have the factorization

$$\begin{array}{ccccccccc} H^i(X) & & & & \xrightarrow{-\cap[X]} & & & & H_{n-i}(X) \\ \downarrow & & & & & & & & \uparrow \\ IH^{\bar{0}}_{n-i}(X) & \longrightarrow & \cdots & \longrightarrow & IH^{\bar{p}}_{n-i}(X) & \longrightarrow & IH^{\bar{q}}_{n-i}(X) & \longrightarrow \cdots \longrightarrow & IH^{\bar{t}}_{n-i}(X) \end{array}$$

Example: M^n a compact manifold-with-boundary ∂M.

$$X^n = M \cup_{\partial M} c(\partial M)$$

$$\xi \in C_i(X) \Rightarrow \dim(|\xi| \cap X_{n-k}) \leqslant i - k + \bar{p}(k)$$

case: $i - n + \bar{p}(n) < -1$. ξ cannot pass through the cone-point c. If $\xi = \partial \xi'$, then ξ' can also not pass through c.

$$\Rightarrow IH^{\bar{p}}_i(X) = H_i(M)$$

case: $i - n + \bar{p}(n) = -1$. $\xi \cap \{c\} = \emptyset$. If $\xi = \partial \xi'$, then ξ' is allowed to touch c. Therefore

$$IH^{\bar{p}}_i(X) = \mathrm{Im}\,(H_i(M) \to H_i(X))$$

case: $i - n + \bar{p}(n) > -1$. Both ξ and ξ' are allowed to pass through c.

$$IH^{\bar{p}}_i(X) = H_i(X)$$

summary:

$$IH_i^{\bar{p}}(X) = \begin{cases} H_i(M) & i < n - \bar{p}(n) - 1 \\ \mathrm{Im}\,(H_i M \to H_i X) & i = n - \bar{p}(n) - 1 \\ H_i(X) & i > n - \bar{p}(n) - 1 \end{cases}$$

Example: $X \rightsquigarrow \mathbb{R} \times X$. From a stratification $\{X_i\}$ of X we get a stratification of $\mathbb{R} \times X$ by setting

$$(\mathbb{R} \times X)_i := \mathbb{R} \times (X_{i-1})$$

$$\xi \in IC_{i-1}^{\bar{p}}(X) \rightsquigarrow \mathbb{R} \times \xi \in IC_i^{\bar{p}}(\mathbb{R} \times X)$$

$$\Rightarrow \text{Suspension map: } IC_*^{\bar{p}}(X) \to IC_*^{\bar{p}}(\mathbb{R} \times X)[1]$$

Lemma 3.14. *This map induces an isomorphism*

$$IH_i^{\bar{p}}(X) \xrightarrow{\cong} IH_{i+1}^{\bar{p}}(\mathbb{R} \times X)$$

Example: $X^4 = (\Sigma T^2) \times S^1$
We compute $IH_*^{\bar{m}}(X), IH_*^{\bar{n}}(X)$

$$T^2 = S_1^1 \times S_2^1, S^1 = \{\mathrm{pt}\} \times S_1$$

Where the point is from the nonsingular part of ΣT^2.
There is one stratum

$$X^4 \supset X^1 = \{N, S\} \times S^1$$

with codimension 3 and link T^2.
We have to decide, whether a cycle bounds, and if it does, if the bounded chain suffices

$$\dim(|\xi_i| \cap X_1) \leqslant i - 3, \dim(|\partial\xi_i| \cap X_1) \leqslant i - 4$$

for $IH_*^{\bar{m}}(X)$ and

$$\dim(|\xi_i| \cap X_1) \leqslant i - 2, \dim(|\partial\xi_i| \cap X_1) \leqslant i - 3$$

for $IH_*^{\bar{n}}(X)$.
If it does, it disappears in the respective intersection-homology-class.

	cycles	$IH_*^{\bar{m}}$	$IH_*^{\bar{n}}$
$* = 4$	$\Sigma(S_1^1 \times S_2^1) \times S^1$	$\Sigma(S_1^1 \times S_2^1) \times S^1$	$\Sigma(S_1^1 \times S_2^1) \times S^1$
$* = 3$	$S_1^1 \times S_2^1 \times S^1, \Sigma S_1^1 \times S^1, \Sigma S_2^1 \times S^1, \Sigma(S_1^1 \times S_2^1)$	$\Sigma(S^1 \times S_2^1)$	$\Sigma(S_1^1 \times S_2^1), \Sigma S_1 \times S^1, \Sigma S_2 \times$
$* = 2$	$S_1^1 \times S_2^1, \Sigma S_1^1, \Sigma S_2^1, S_1^1 \times S^1 S_2^1 \times S^1$	$S_1^1 \times S^1, S_2^1 \times S^1$	$\Sigma S_1, \Sigma S_2$
$* = 1$	S_1^1, S_2^1, S^1	S_1^1, S_2^1, S^1	S^1
$* = 0$	pt	pt	pt

3.1 The intersection homology of a cone

Let X^{k-1} be a compact PL-stratified pseudomanifold.

$$(c^\circ X)_i := c^\circ(X_{i-1}), i > 0$$

$$(c^\circ X)_0 := \{c\}$$

Let $\xi \in IC_{i-1}^{\bar{p}}(X)$. We can form $c^\circ\xi \in C_i(c^\circ X)$.
But it is not generally true, that $c^\circ\xi \in IC_i^{\bar{p}}(c^\circ X) \subset C_i(c^\circ X)$.
If $i > k - \bar{p}(k)$, then $c^\circ\xi \in IC_i^{\bar{p}}(c^\circ X)$.
If $i = k - \bar{p}(k)$, then

$$c^\circ\xi \in IC_i^{\bar{p}}(c^\circ X) \Leftrightarrow \partial\xi = 0$$

If $i < k - \bar{p}(k)$, then

$$c^\circ\xi \in IC_i^{\bar{p}}(c^\circ X) \Leftrightarrow \xi = 0$$

This suggests that coning should be viewed as a map of chaincomplexes:

$$
\begin{array}{ccc}
\cdots & \xrightarrow{c^\circ} & \cdots \\
\downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} \\
IC_{k-\bar{p}(k)+1}(X) & \xrightarrow{c^\circ} & IC_{k-\bar{p}(k)+2}(c^\circ X) \\
\downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} \\
IC_{k-\bar{p}(k)}(X) & \xrightarrow{c^\circ} & IC_{k-\bar{p}(k)+1}(c^\circ X) \\
\downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} \\
\ker \partial & \xrightarrow{c^\circ} & IC_{k-\bar{p}(k)}(c^\circ X) \\
\downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} \\
0 & \xrightarrow{c^\circ} & IC_{k-\bar{p}(k)-1}(c^\circ X) \\
\downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} \\
\cdots & \xrightarrow{c^\circ} & \cdots
\end{array}
$$

We have constructed a map

$$\tau_{\geqslant k-\bar{p}(k)-1} IC_*^{\bar{p}}(X) \xrightarrow{c^\circ} IC_*^{\bar{p}}(c^\circ X)[1]$$

Theorem 3.15. *This chain map induces an isomorphism on homology groups.*

Lemma 3.16. *If α is a PL chain in $\mathbb{R}_{\geqslant 0} \times Z$ and $\partial\alpha = 0$, then there exist a PL-chain β, such that $\alpha = \partial\beta$.*

Proof.

$$\pi_1 : \mathbb{R}_+ \times Z \to \mathbb{R}_+, \pi_2 : \mathbb{R}_+ \times Z \to Z$$

$$\mathbb{R}_{\geqslant 0} \times |\alpha| \xrightarrow{f} \mathbb{R}_+ \times Z$$

$$f(t,x) := (t + \pi_1(x), \pi_2(x))$$

$$\beta := f_*(\mathbb{R}_{\geqslant 0} \times \alpha) \Rightarrow \partial\beta = \alpha$$

□

Proof. of thm. 3.15
c° is injective as a chain map. Exact sequence:

$$0 \to \tau_{\geqslant k-\bar{p}(k)-1} IC_*^{\bar{p}}(X) \xrightarrow{c^\circ} IC_*^{\bar{p}}(c^\circ X)[1] \to \frac{IC_*^{\bar{p}}(c^\circ X)}{c^\circ \tau_{\geqslant k-\bar{p}(k)-1} IC_*^{\bar{p}}(X)} \to 0$$

We will show, that $\frac{IC_*^{\bar{p}}(c^\circ X)}{c^\circ \tau_{\geqslant k-\bar{p}(k)-1} IC_*^{\bar{p}}(X)}$ is acyclic.
Given $\xi \in IC_i^{\bar{p}}(c^\circ X)$, such that $\partial\xi = c^\circ\gamma$, where $\gamma \in (\tau_{k-\bar{p}(k)-1} IC_*^{\bar{p}}(X))_{i-2}$.
We will construct η, such that $\xi - c^\circ\eta$ is a cycle that bounds.

$$\exists \epsilon > 0\colon |\xi| \cap N_\epsilon = c^\circ\eta, \partial\eta = \gamma$$

$$\partial(\xi - c^\circ\eta) = \partial\xi - \partial c^\circ\eta = c^\circ\gamma - c^\circ\partial\eta = c^\circ\gamma - c^\circ\gamma = 0$$

So $\xi - c^\circ\eta$ is a cycle.
And:

$$|\xi - c^\circ\eta| \subset \mathbb{R}_{\geqslant 0} \times X$$

With lemma 3.16 it follows, that

$$\xi - c^\circ\eta = \partial\beta$$

(check: This β is an intersection chain) □

Theorem 3.15 implies

$$IH_i^{\bar{p}}(c^\circ X) = \begin{cases} IH_{i-1}^{\bar{p}}(X) & i \geqslant k - \bar{p}(k) \\ 0 & i < k - \bar{p}(k) \end{cases}$$

Corollary 3.17. *$IH_*^{\bar{p}}(-)$ is not a homotopy invariant.*

Example: Recall that a point $x \in X_{n-k} \backslash X_{n-k-1}$ in a stratified pseudomanifold has a neighbourhood of the form $\mathbb{R}^{n-k} \times c^\circ(L^{k-1})$ (”distinguished neighbourhood“)

$$\tau_{k-\bar{p}(k)-1} IC^{\bar{p}}_\bullet(L) \overset{\text{susp.}^{n-k} \circ c^\circ}{\to} IC^{\bar{p}}_\bullet(\mathbb{R}^{n-k} \times c^\circ L)[n-k+1]$$

is a homology-isomorphism (”quasi-isomorphism“).
In particular,

$$IH^{\bar{p}}_i(\mathbb{R}^{n-k} \times c^\times L) \cong \begin{cases} IH^{\bar{p}}_{i-(n-k+1)}(L) & i \geqslant n - \bar{p}(k) \\ 0 & i < n - \bar{p}(k) \end{cases}$$

4 Sheaf theory

Definition 4.1 (presheaf)**.** *A* ***presheaf*** *A on a topological space X is an assignment*

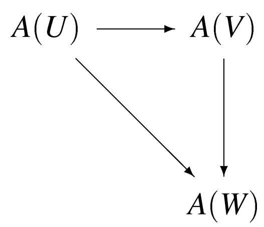
$$U \underset{\text{open}}{\subset} X \mapsto A(U)$$

from the open sets in X to abelian groups, such that there are restriction homomorphisms

$$\forall U, V \underset{\text{open}}{\subset} U \colon A(U) \to A(V)$$

such that

1. $A(\varnothing) = 0$
2. $A(U) \overset{\text{id}}{\to} A(U)$
3. *The following diagram commutes for* $W \underset{\text{open}}{\subset} V \underset{\text{open}}{\subset} U$



The $A(U)$ *are called* ***section groups****.*

Definition 4.2 (sheaf)**.** *A* ***sheaf*** $\underline{A}$ *on X is a topological space with a continous map*

$$\pi \colon \underline{A} \to X,$$

such that for each $x \in X$

$$\pi^{-1}(x) =: \underline{A}_x$$

(we call this a ***stalk****) is an abelian group and the group operations are continous, and* π *is a local homeomorphism.*

Definition 4.3. *Let $\underline{A}$ bei a sheaf on X and $Y \subset X$ a subspace.*
*A **section** of $\underline{A}$ over Y is a continous map*

$$s\colon Y \to A$$

such that $\pi \circ s = \mathrm{id}_Y$.

$$\Gamma(Y, \underline{A}) := \{ \text{ sections } s\colon Y \to \underline{A}\}$$

We can assign a presheaf to $\underline{A}$ by

$$\underline{A} \rightsquigarrow (U \mapsto \Gamma(U, \underline{A}))$$

the **presheaf of sections**.
Conversely, one can associate a sheaf to a given presheaf A:
We define the stalks to be

$$x \in X : \underline{A}_x := \varinjlim_{U \underset{\text{open}}{\subset} X, x \in U} A(U),$$

form the sheaf by

$$\underline{A} := \bigcup_{x \in X} \underline{A}_x$$

and get a projection by

$$\pi\colon \underline{A} \to X$$

$$\underline{A}_x \mapsto \{x\}$$

and a basis for the topology by

$$U \underset{\text{open}}{\subset} X, s \in A(X)\colon \{s_x \in \underline{A}_x \mid x \in U\}$$

where s_x is the equivalence class of x under the direct limit.
This is called the **sheafification** of A and we write it as $\underline{\mathrm{sheaf}}(A)$.

Definition 4.4 (Homomorphism)**.** *A **homomorphism** $f\colon \underline{A} \to \underline{B}$ of sheaves over X is a continous map which preserves stalks, i.e.*

$$f(\underline{A}_x) \subset \underline{B}_x$$

and restricts, for all $x \in X$ to group homomorphisms

$$f\colon \underline{A}_x \to \underline{B}_x$$

*f is a **monomorphism** (epimorphism, isomorphism,...), if f_x is a monomorphism (epimorphism, isomorphism,...) for all x.*

Let $\underline{A}$ be any sheaf. For every $a \in \underline{A}$ choose a local section s through a. By this you get an isomorphism

$$\underline{A} \xrightarrow{\cong} \underline{\text{sheaf}}(U \mapsto \Gamma(U, \underline{A}))$$
$$a \mapsto s_{\pi(a)}$$

Conversely let A be any presheaf. The map

$$A(U) \to \Gamma(U, \underline{\text{sheaf}}(A))$$
$$s \mapsto (x \in U \mapsto s_x)$$

is **not** generally an isomorphism.
The presheaf of sections of an actual sheaf satisfies a **gluing property** (G): Given any open subset $U \underset{\text{open}}{\subset} X$, any open covering $U = \bigcup_\alpha U_\alpha$ and $s_\alpha \in A(U_\alpha)$, such that

$$s_\alpha|_{U_\alpha \cap U_\beta} = s_\beta|_{U_\alpha \cap U_\beta}$$

for $U_\alpha \cap U_\beta \neq \emptyset$,

$$\exists! s \in A(U) \colon s|_{U_\alpha} = s_\alpha, \forall \alpha$$

If A satisfies (G), then the canonical map from above **is** an isomorphism.
Summary:

Proposition 4.5. *There is a one-to-one correspondence between sheaves on X and presheaves on X, that satisfy (G).*

4.1 Sheaf cohomology

Definition 4.6. *A **resolution** of a sheaf $\underline{A}$ is an exact sequence of sheaves*

$$0 \to \underline{A} \overset{\mathrm{d}^{-1}}{\hookrightarrow} \underline{K}^0 \overset{\mathrm{d}^0}{\to} \underline{K}^1 \overset{\mathrm{d}^1}{\to} \underline{K}^2 \overset{\mathrm{d}^2}{\to} \dots$$

*A **morphism of resolutions** $\underline{A} \to \underline{K}^\bullet$, $\underline{B} \to \underline{L}^\bullet$ is a commutive diagram*

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \underline{A} & \longrightarrow & \underline{K}^0 & \longrightarrow & \underline{K}^1 & \longrightarrow & \underline{K}^2 & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \phi^0} & & \downarrow{\scriptstyle \phi^1} & & \downarrow{\scriptstyle \phi^2} & & \\
0 & \longrightarrow & \underline{A} & \longrightarrow & \underline{K}^0 & \longrightarrow & \underline{K}^1 & \longrightarrow & \underline{K}^2 & \longrightarrow & \cdots
\end{array}$$

*A **homotopy** of homomorphisms of resolutions is a sequence of sheaf homomorphisms s, such that* $\mathrm{d}s + s\,\mathrm{d} = \phi - \psi$.

Definition 4.7. *A sheaf $\underline{I}$ is **injective**, if for every monomorphism $\underline{A} \hookrightarrow \underline{B}$ and homomorphism $f\colon \underline{A} \to \underline{I}$, there is an extension $\underline{B} \to \underline{I}$, such that the following diagram commutes:*

$$\begin{array}{ccccc}
0 & \longrightarrow & \underline{A} & \longrightarrow & \underline{B} \\
 & & \downarrow & \swarrow & \\
 & & \underline{I} & &
\end{array}$$

Basic constructions: X, Y topological spaces, $f\colon X \to Y$ continuous.

1. Let $\underline{A}$ be a sheaf on X. $U \mapsto \Gamma(f^{-1}(U), \underline{A})$ is a presheaf satisfying (G), so a sheaf, called the **pushforward** or **direct image** sheaf of $\underline{A}$. Notation: $f_*\underline{A}$ (a sheaf on Y).

2. Lem $\underline{B}$ be a sheaf on Y. We can form the pullback:

$$f^*\underline{B} := \{(x,b) \in X \times B \mid f(x) = \pi(b)\}$$

$$(f^*\underline{B})_x = \underline{B}_{f(x)}$$

   $\Rightarrow f^*\underline{B}$ is a sheaf over X, the **pullback** of $\underline{B}$.

Lemma 4.8. *The product of injective sheaves is injective.*

Proposition 4.9. *Every sheaf can be embedded into an injective sheaf (even canonically).*

Proof. Let $\underline{A}$ be any sheaf on X, $x \in X$.
We can then embed $\underline{A}_x$ as an abelian group into an injective group I_x. Let $\underline{I}_x$ be the extension by zero of I_x.
By lemma 4.8,

$$\underline{I} := \prod_{x \in X} \underline{I}_x$$

is an injective sheaf, because all $\underline{I}_x$ are injective sheafs.

$$(\underline{A} \to \underline{I}_x) \rightsquigarrow (\underline{A} \to \underline{I})$$

□

Proposition 4.10. *Every sheaf has a (canonical) injective resolution.*

Proof. According to proposition 4.9 we get injective $\underline{I}^k$, such that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \underline{A} & \longrightarrow & \underline{I}^0 & \xrightarrow{\mathrm{d}^0} & & \underline{I}^1 & \xrightarrow{\mathrm{d}^1} \\ \mathrm{d}^{-1} & & & & & \searrow & & \nearrow & \\ & & & & & & \underline{I}^0/\mathrm{Im}\ \mathrm{d}^{-1} & & \end{array}$$

is exact. □

Proposition 4.11. *$f\colon \underline{A} \to \underline{B}$ can be extendet to a morphism of resolutions.*

Proof. Per diagram-hunt. □

By similar extension arguments, any two such extensions of f are homotopic.

Definition 4.12 (Sheaf cohomology)**.** *Let $\underline{A}$ be a sheaf on X and $I^\bullet$ be an injective resolution of $\underline{A}$.*

$$H^i(X; \underline{A}) := H^i(\Gamma(X, I^\bullet))$$

Remark: Thus, $H^i(X, \underline{A})$ measures the extent to which $\Gamma(X, -)$ is not an **exact** functor.
Remark: $\Gamma(X, -)$ is left exact.
Remark: $H^*(X; \underline{A})$ is well-defined.

Definition 4.13. *A sheaf $\underline{A}$ is called* ***acyclic****, if*

$$\forall i > 0\colon H^i(X, A) = 0$$

Example: Injective sheaves are acyclic:

$$0 \to \underline{I} \xrightarrow{\text{id}} \underline{I} \to 0 \to 0 \to ...$$

$$\Rightarrow H^i(X, \underline{I}) = 0, \forall i > 0$$

Example: (Assume X paracompact) A sheaf $\underline{A}$ is called **soft**, if forall closed $K \subset X$

$$\Gamma(X, \underline{A}) \to \Gamma(K, \underline{A})$$

is surjective.
Soft sheaves are acyclic.

Definition 4.14. *If $\underline{K}^\bullet$ is a resolution of a sheaf $\underline{A}$ by acyclic sheaves, then*

$$H^*(X; \underline{A}) \cong H^*\Gamma(X, \underline{K}^\bullet)$$

So we can use acyclic resolutions, which occur more naturally than injective ones, to compute sheaf cohomology.
Application: Let M be a smooth manifold. We compute the sheaf cohomology

$$H^*(M; \underline{\mathbb{R}}_M)$$

Where $\underline{\mathbb{R}}_M$ is a constant sheaf. We can map $U \underset{\text{open}}{\subset} X$ as of

$$U \mapsto \Omega^i(U)$$

where $\Omega^i(U)$ are smooth differential i-forms on U. This presheaf satisfies (G), so is a sheaf $\underline{\Omega}^i$.
Since the exterior derivative

$$\mathrm{d}\colon \Omega^i(U) \to \Omega^{i+1}(U)$$

commutes with restricton-maps, it induces a morphism of sheaves

$$\mathrm{d}\colon \underline{\Omega}^i \to \underline{\Omega}^{i+1}$$

such that $\mathrm{d}^2 = 0$.

$$\underline{\Omega^{i-1} \xrightarrow{\mathrm{d}} \Omega^i \xrightarrow{\mathrm{d}} \Omega^{i+1}}$$

is an exact sequence of sheaves by the Poincaré lemma.
Therefore

$$0 \to \underline{\mathbb{R}}_M \xrightarrow{r \mapsto f \equiv r} \underline{\Omega}^0 \xrightarrow{\mathrm{d}} \underline{\Omega}^1 \xrightarrow{\mathrm{d}} \underline{\Omega}^2 \xrightarrow{\mathrm{d}} ...$$

is a resolution of $\underline{\mathbb{R}}_M$. All $\underline{\Omega}^i$ are soft.

$$H^*(M; \underline{\mathbb{R}}_M) = H^*\Gamma(M; \underline{\Omega}^i) = H^*(\Omega^\bullet(M)) = H^*(M)$$

So we can compute the sheaf cohomology $H^*(M; \underline{\mathbb{R}}_M)$ by computing the de-Rham-cohomology of M.

4.2 Complexes of sheaves

Definition 4.15. *A (differential)* ***complex of sheaves*** $\underline{A}^\bullet$ *is a sequence* $(\underline{A}^i)_{i\in\mathbb{Z}}$ *of sheaves* $\underline{A}^i$*, together with sheaf homomorphisms*

$$\mathrm{d}^i\colon \underline{A}^i \to \underline{A}^{i+1}$$

such that

$$\forall i \in \mathbb{Z}\colon\ \mathrm{d}^{i+1} \circ \mathrm{d}^i = 0$$

$$\underline{H}^i(\underline{A}^\bullet) := \frac{\underline{\ker \mathrm{d}}}{\underline{\operatorname{Im}}\ \mathrm{d}} = \underline{\text{sheaf}}\left(U \mapsto H^i\Gamma(U;\underline{A}^\bullet)\right)$$

is called the **cohomology sheaf** or **derived sheaf**.
Say $\underline{A}^\bullet$ is bounded below. We can then form injective resolutions of the $\underline{A}^i$ and get

$$\begin{array}{ccccccccc}
\cdots & & \cdots & & \cdots & & \cdots & & \\
\uparrow{\scriptstyle \mathrm{d}^2} & & \uparrow{\scriptstyle \mathrm{d}} & & \uparrow{\scriptstyle \mathrm{d}} & & \uparrow{\scriptstyle \mathrm{d}} & & \\
\underline{A}^2 & \longrightarrow & \underline{I}^{0,2} & \xrightarrow[\delta]{} & \underline{I}^{1,2} & \xrightarrow[\delta]{} & \underline{I}^{2,2} & \xrightarrow[\delta]{} & \cdots \\
\uparrow{\scriptstyle \mathrm{d}^2} & & \uparrow{\scriptstyle \mathrm{d}} & & \uparrow{\scriptstyle \mathrm{d}} & & \uparrow{\scriptstyle \mathrm{d}} & & \\
\underline{A}^1 & \longrightarrow & \underline{I}^{0,1} & \xrightarrow[\delta]{} & \underline{I}^{1,1} & \xrightarrow[\delta]{} & \underline{I}^{2,1} & \xrightarrow[\delta]{} & \cdots \\
\uparrow{\scriptstyle \mathrm{d}^2} & & \uparrow{\scriptstyle \mathrm{d}} & & \uparrow{\scriptstyle \mathrm{d}} & & \uparrow{\scriptstyle \mathrm{d}} & & \\
\underline{A}^0 & \longrightarrow & \underline{I}^{0,0} & \xrightarrow[\delta]{} & \underline{I}^{1,0} & \xrightarrow[\delta]{} & \underline{I}^{2,0} & \xrightarrow[\delta]{} & \cdots
\end{array}$$

Such a diagram is called a **double complex**, with a vertical differential d, a horizontal differental δ, such that

$$\mathrm{d}^2 = \delta^2 = \mathrm{d}\delta + \delta\,\mathrm{d} = 0$$

Via

$$\underline{I}^k := \bigoplus_{i+j=k} \underline{I}^{i,j}$$

we get a simple complex with the differential $\mathrm{d} + \delta$.

Definition 4.16. *A* ***resolution of a complex of sheaves*** $\underline{A}^\bullet$ *is a* ***quasi-isomorphism*** $f\colon \underline{A}^\bullet \to \underline{K}^\bullet$ *(i.e. f induces isomorphisms $\underline{H}^i(\underline{A}^\bullet) \to \underline{H}^i(\underline{K}^\bullet), \forall i$).*

We have seen that every complex $\underline{A}^\bullet$ has an injective resolution $\underline{A}^\bullet \to \underline{I}^\bullet$.

Definition 4.17. *The* ***hypercohomology groups*** *of X with coefficients in a complex of sheaves $\underline{A}^\bullet$ are*

$$\mathscr{H}^i(X;\underline{A}^\bullet) := H^i\Gamma(X;\underline{I}^\bullet)$$

where $\underline{A}^\bullet \to \underline{I}^\bullet$ is an injective resolution of $\underline{A}^\bullet$.

Remark: This can also be computed from a resolution $\underline{A}^\bullet \to \underline{K}^\bullet$ of $\underline{A}^\bullet$ by soft sheaves $\underline{K}^i$.
Standard constructions:

$$\tau_{\leqslant n}\underline{A}^\bullet := \ldots \to \underline{A}^{n-2} \xrightarrow{\mathrm{d}} \underline{A}^{n-1} \to \underline{\ker}\,\mathrm{d}^n \to 0 \to 0 \to \ldots$$

$$\tau_{\geqslant n}\underline{A}^\bullet := \ldots \to 0 \to 0 \to \operatorname{coker}\mathrm{d}^{n-1} \to \underline{A}^{n+1} \to \underline{A}^{n+2} \to \ldots$$

$$\Rightarrow \underline{H}^i(\tau_{\leqslant n}\underline{A}^\bullet) = \begin{cases} \underline{H}^i(\underline{A}^\bullet) & i \leqslant n \\ 0 & i > n \end{cases}$$

$$\underline{H}^i(\tau_{\geqslant n}\underline{A}^\bullet) = \begin{cases} 0 & i < n \\ \underline{H}^i(\underline{A}^\bullet) & i \geqslant n \end{cases}$$

$$\tau_{\leqslant n}\underline{A}^\bullet \xrightarrow{\text{inc}} \underline{A}^\bullet \xrightarrow{\text{quot}} \tau_{\geqslant n}\underline{A}^\bullet$$

$$\underline{A}^\bullet[n])^i := \underline{A}^{i+n},\ \mathrm{d}^i_{\underline{A}^\bullet[n]} = (-1)^n\,\mathrm{d}^{i+n}_{\underline{A}^\bullet}$$

Also here: $\underline{A}^\bullet \oplus \underline{B}^\bullet, \underline{A}^\bullet \otimes \underline{B}^\bullet, \underline{\mathrm{Hom}}^\bullet(\underline{A}^\bullet, \underline{B}^\bullet), f_*\underline{A}^\bullet, f^*\underline{B}^\bullet$
Complexes of sheaves form an abelian (i.e. have kernels, cokernels, images, coimages, exact sequences, etc. with the usual properties) category $C(X)$.

4.3 The homotopy category

Definition 4.18. *The* ***homotopy category*** $K(X)$ *has objects*

$$\mathrm{Ob}K(X) := \mathrm{Ob}C(X)$$

and morphisms

$$Hom_{K(X)}(\underline{A}^\bullet, \underline{B}^\bullet) := \{[f] \mid f \in Hom_{C(X)}(\underline{A}^\bullet, \underline{B}^\bullet)\}$$

where $[f]$ *denotes the homotopy-class of* f.

$$[f] \circ [g] := [f \circ g]$$

is well-defined, so $K(X)$ forms indeed a category.

$$[f] + [g] := [f + g]$$

is also well-defined, so $K(X)$ is an additive category.
If $\underline{A}^\bullet$ and $\underline{B}^\bullet$ are homotopy equivalent, then $\underline{A}^\bullet \cong \underline{B}^\bullet$ in $K(X)$.
Problem: $K(X)$ is not in general an abelian category.

Example: $X = \{p\}$

$$
\begin{array}{ccccccccc}
 & & \cdots & & \cdots & & \cdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \cdots & & \cdots & & \cdots & &
\end{array}
$$

We have a short exact sequence of complexes

$$0 \to X^\bullet \to Y^\bullet \to Z^\bullet \to 0$$

$$Y^\bullet \simeq 0$$

We have a natural functor $C(X) \xrightarrow{F} K(X)$.
If the resulting sequence was to be exact

$$0 \to F(X^\bullet) \to F(Y^\bullet) \cong 0 \to F(Z^\bullet) \to 0$$

But

$$H^\bullet(F(X^\bullet)) = H^\bullet(X) \neq 0$$

which is a contradiction.
Solution: ”Distinguished triangles“, substitute for exact sequences.

Definition 4.19. *Let $f\colon \underline{A}^\bullet \to \underline{B}^\bullet$ be a morphism in $C(X)$. The (algebraic)* ***mapping cone*** *of f, $\underline{C}^\bullet(f)$, is the complex given by*

$$C^n(f) := \underline{A}^{n+1} \oplus \underline{B}^n$$

$$\mathrm{d}^n_{C^\bullet(f)} = \begin{pmatrix} \mathrm{d}_{\underline{A}} & 0 \\ f^{n+1} & \mathrm{d}_{\underline{B}} \end{pmatrix}$$

Remark: A continuous map $f\colon X \to Y$ between topological spaces has a (topological) mapping cone: f induces a map $f^{\#}\colon C^*(Y) \to C^*(X)$. Then the (reduced) cohomology of the topological mapping cone is

$$H^*(C^\bullet(f^{\#}))$$

There are canonical maps

$$\underline{A}^\bullet \overset{\text{inc}}{\to} \underline{C}^\bullet(f) \overset{\text{proj}}{\to} \underline{A}^\bullet[1]$$

Definition 4.20. *Any sequence in $K(X)$ isomorphic to a sequence of the form*

$$\underline{A}^\bullet \overset{f}{\to} \underline{B}^\bullet \overset{\text{inc}}{\to} \underline{C}^\bullet(f) \overset{\text{proj}}{\to} \underline{A}^\bullet[1] \overset{f[1]}{\to} \underline{B}^\bullet[1] \to \ldots$$

*is called a **distinguished** triangle.*

Notation: We usually write

$$\underline{A}^\bullet \overset{f}{\to} \underline{B}^\bullet \to \underline{C}^\bullet(f) \overset{+1}{\to}$$

or

$$\begin{array}{ccc} \underline{A}^\bullet & \xrightarrow{\quad f \quad} & \underline{B}^\bullet \\ {\scriptstyle [1]}\nwarrow & & \swarrow \\ & \underline{C}^\bullet(f) & \end{array}$$

Properties:

1. $\underline{A}^\bullet \overset{\text{id}}{\to} \underline{A}^\bullet \to 0 \overset{+1}{\to}$ is distinguished
2. If
$$\underline{A}^\bullet \overset{a}{\to} \underline{B}^\bullet \overset{b}{\to} \underline{C}^\bullet \overset{+1,c}{\to}$$
is distinguished, so is
$$\underline{B}^\bullet \overset{b}{\to} \underline{C}^\bullet \overset{c}{\to} \underline{A}^\bullet[1] \overset{+1,-a[1]}{\to}$$
3. Given
$$\begin{array}{ccccccccc} \underline{A}^\bullet & \longrightarrow & \underline{B}^\bullet & \longrightarrow & \underline{C}^\bullet & \longrightarrow & \underline{A}^\bullet[1] & \longrightarrow & \cdots \\ \downarrow a & & \downarrow b & & & & \downarrow a[1] & & \\ \underline{\hat{A}}^\bullet & \longrightarrow & \underline{\hat{B}}^\bullet & \longrightarrow & \underline{\hat{C}}^\bullet & \longrightarrow & \underline{\hat{A}}^\bullet[1] & \longrightarrow & \cdots \end{array}$$
Then there exists a (not unique) morphism
$$c\colon \underline{C}^\bullet \to \underline{\hat{C}}^\bullet$$
such that the diagram commutes.

If $\underline{A}^\bullet \to \underline{B}^\bullet \to \underline{C}^\bullet \xrightarrow{+1}$ is a distinguished triangle, then

1. $\underline{H}^\bullet(\underline{A}^\bullet) \to \underline{H}^\bullet(\underline{A}^\bullet) \to \underline{H}^\bullet(\underline{C}^\bullet) \to \underline{H}^{\bullet+1}(\underline{A}^\bullet) \to ...$ is exact
2. $\mathscr{H}^\bullet(X; \underline{A}^\bullet) \to \mathscr{H}^\bullet(X; \underline{B}^\bullet) \to \mathscr{H}^\bullet(X; \underline{C}^\bullet) \to \mathscr{H}^{\bullet+1}(X; \underline{A}^\bullet) \to ...$ is exact

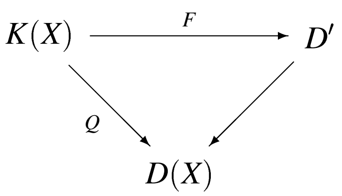
Problem: A short exact sequence $0 \to \underline{A}^\bullet \xrightarrow{a} \underline{B}^\bullet \xrightarrow{b} \underline{C}^\bullet \xrightarrow{c} 0$ should induce a distinguished triangle $\underline{A}^\bullet \xrightarrow{a} \underline{B}^\bullet \xrightarrow{b} \underline{C}^\bullet \xrightarrow{+1}$.
Attempt: $\underline{A}^\bullet \xrightarrow{a} \underline{B}^\bullet \to \underline{C}^\bullet(a) \xrightarrow{+1}$

$$\underline{C}^n(a) = \underline{A}^{n+1} \oplus \underline{B}^n \to \underline{C}^n$$

$$(\alpha, \beta) \mapsto b(\beta)$$

$$\rightsquigarrow \underline{C}^\bullet(a) \xrightarrow{f} \underline{C}^\bullet$$

The 5-Lemma implies, that f is a quasi-isomorphism.
However, f is not a homotopy-equivalence in genereal (It is, if b splits).
This problem leads to the notion of **derived categories**: There exists a category $D(X)$ and a functor $Q\colon K(X) \to D(X)$, such that:
Given any (additive) category D' and a functor $F\colon K(X) \to D'$, such that F of a quasi-isomorphism is an isomorphism in D', there is a unique functor $D' \to D(X)$, such that the following diagram commutes:



$D(X)$ solves the above problem.
But there is a new problem: A functor on the category of sheaves

$$F\colon \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$$

does not in general induce a functor

$$F\colon D(X) \to D(Y)$$

Let $\underline{A}^\bullet \in C(X)$ be an acyclic complex, such that $F(\underline{A}^\bullet)$ is not acyclic (e.g. $Y = \{p\}, \mathrm{Sh}(\{p\})$ is just the category of abelien groups, $F = \Gamma(X, -)$). Then $\underline{A}^\bullet \to 0^\bullet$ is a quasi-isomorphism, but $F(\underline{A}^\bullet) \to F(0^\bullet) = 0^\bullet$ is not a quasi-isomorphism.
Solution: Derived Functors

Definition 4.21. *Let F be a functor from* $\mathrm{Sh}(X)$. *The **right derived** Functor is defined as*

$$RF(\underline{A}^\bullet) := F(\underline{I}^\bullet)$$

where $\underline{A}^\bullet \to \underline{I}^\bullet$ is an injective resolution.

Lemma 4.22. *If $f\colon \underline{I}^\bullet \to \underline{A}^\bullet$ is a quasi-isomorphism, with $\underline{I}^\bullet$ injective, then f is a homotopy-equivalence.*

So, if $\underline{I}^\bullet \xrightarrow{f} \underline{J}^\bullet$ is a quasi-isomorphism, then $F(\underline{I}^\bullet) \xrightarrow{F(f)} F(\underline{J}^\bullet)$ is also a quasi-isomorphism.
Example: $f\colon X \to Y \rightsquigarrow Rf_*$
Example: $Y = \{p\}\colon f_* = \Gamma(X;-) \rightsquigarrow R\Gamma(X,-)$

$$\Rightarrow \mathscr{H}^\bullet(X, \underline{A}^\bullet) = H^\bullet(R\Gamma(X; \underline{A}^\bullet))$$

Example: But: $f^*\colon D(Y) \to D(X)$ (i.e. $Rf^* = f^*$), because f^* is exact ($(f^*\underline{A})_x = \underline{A}_{f(x)}$ and exactnes can be checked stalkwise).

5 The sheafification of the intersection chain complex

Let X be a PL-stratified (oriented) pseudomanifold.

$$U \underset{\text{open}}{\subset} X \mapsto IC_i^{\bar{p}}(U)$$

is a presheaf, since we work with Borel-Moore chains.
This presheaf satisfies (G), and so is a **sheaf** $\underline{IC}_{\bar{p}}^{-i}(X)$ on X.
The boundary maps

$$IC_i^{\bar{p}}(U) \to IC_{i-1}^{\bar{p}}(U)$$

induce differentials

$$\mathrm{d}\colon \underline{IC}_{\bar{p}}^{-i}(X) \to \underline{IC}_{\bar{p}}^{-i+1}(X)$$

$$\Gamma(U;\underline{IC}^\bullet_{\bar{p}}(X)) = IC^{\bar{p}}_{-\bullet}(U)$$

Lemma 5.1. *Let X be a paracompact topological space, K be a closed subset of X and $\underline{A}$ be a sheaf on X. Then*

$$\Gamma(K;\underline{A}) = \varinjlim_{\substack{U\supset X\\ \text{open}}} \Gamma(U;\underline{A})$$

Theorem 5.2. *$\underline{IC}^\bullet_{\bar{p}}(X)$ is soft.*

Proof. Let $K \subset X$ be closed, $\xi \in \Gamma(K;\underline{IC}^\bullet_{\bar{p}}(X)$. Lemma 5.1 $\Rightarrow$ ξ is represented by some $\hat{\xi} \in \Gamma(U,\underline{IC}^\bullet_{\bar{p}}(X) = IC^{\bar{p}}_{-\bullet}(U)$, where U is an open neighbourhood of K.
Triangulate U by T so that $\hat{\xi}$ is simplicial and all strata are subcomplexes.
We take every vertex, which bounds a simplex, that intersects K nontrivially. We then take the stars of all the stars of such vertices in respect to the first barycentric subdivision T' and call the union N.
Then $\hat{\xi} \cap N \in IC^{\bar{p}}_\bullet(X)$, because $\mathrm{St}(v,T')$ is transverse to the strata, so it is in $IC^{\bar{0}}$, therefore $\hat{\xi} \cap \mathrm{St}(v,T') \in IC^{\bar{p}}_\bullet$.
In the direct limit, $\hat{\xi} \cap N$ represents ξ. □

Corollary 5.3.

$$\mathscr{H}^{-i}(X;\underline{IC}^\bullet_{\bar{p}}(X)) = IH^{\bar{p}}_i(X)$$

Proof.

$$\begin{aligned}
\mathscr{H}^{-i}(X;\underline{IC}^\bullet_{\bar{p}}(X)) &= H^{-i}\Gamma(X;\underline{IC}^\bullet_{\bar{p}}(X)) \\
&= H^{-i}IC_{\bar{p}}(X) \\
&= H_i IC^{\bar{p}}_\bullet(X) \\
&= IH^{\bar{p}}_i(X)
\end{aligned}$$

□

Stalks of $IC^\bullet_{\bar{p}}(X)$: Let $x \in X_{n-k}\backslash X_{n-k-x}$.

$$\begin{aligned}
\underline{H}^{-i}(\underline{IC}^\bullet_{\bar{p}})_x &= \varinjlim_{U\ni x} H^{-i}\Gamma(U;\underline{IC}^\bullet_{\bar{p}}) \\
&= \varinjlim_{U\ni x} IH^{\bar{p}}_i(U) \\
&= \varinjlim_{\text{disting.}U\ni x} IH^{\bar{p}}_i(U) \\
&= \lim_{\epsilon\to 0} IH^{\bar{p}}_i((-\epsilon,\epsilon)^{n-k}\times c^\circ_\epsilon(L_x)) \\
&= \begin{cases} IH_{i-(n-k+1)}(L_x) & i \geqslant n-\bar{p}(k) \\ 0 & i < n-\bar{p}(k) \end{cases}
\end{aligned}$$

$\Rightarrow$ cohomology-stalk **vanishing**.

$$\underline{H}^{-i}(\underline{IC}^{\bullet}_{\bar{p}})_x = IH_i^{\bar{p}}(U_x)$$

where U_x is a small distinguished neighbourhood of x.
Notation: $U_k := X\backslash X_{n-k}, k \geqslant 2$, $i_k\colon U_k \hookrightarrow U_{k+1}$

$$\begin{aligned}
\underline{H}^{-i}(i_{k*}\underline{IC}^{\bullet}_{\bar{p}}|_{U_k}) &= \varinjlim_{U\ni x} H^{-i}\Gamma(U; i_{k*}\underline{IC}^{\bullet}_{\bar{p}}|_{U_k}) \\
&= \varinjlim_{U\ni x} H^{-i}\Gamma(U\cap U_k; \underline{IC}^{\bullet}_{\bar{p}}) \\
&= \lim_{\epsilon\to 0} IH_i^{\bar{p}}(U_\epsilon \cap U_k) \\
&= \lim_{\epsilon\to 0} IH_i^{\bar{p}}((-\epsilon,\epsilon)^{n-k} \times ((c^\circ L_x)\backslash\{c\})) \\
&= \lim_{\epsilon\to 0} IH_i^{\bar{p}}((-\epsilon,\epsilon)^{n-k+1} \times L_x) \\
&= IH^{\bar{p}}_{i-(n-k+1)}(L_x)
\end{aligned}$$

$$\underline{H}^{-i}(i_{k*}i_k^*(IC^{\bullet}_{\bar{p}}|_{U_{k+1}})) = IH_i^{\bar{p}}(U_x\cap U_k)$$

We therefore get from the restriction

$$IH_i^{\bar{p}}(U_x) \to IH_i^{\bar{p}}(U_x\cap U_k)$$

a natural map

$$\underline{H}^{-i}(IC^{\bullet}_{\bar{p}}|_{U_{k+1}})_x \to \underline{H}^{-i}(i_{k*}i_k^*(IC^{\bullet}_{\bar{p}}))_x$$

Abstractly: $f\colon X\to Y, \underline{A}\in \mathrm{Sh}(X), \underline{B}\in\mathrm{Sh}(Y)$

$$\mathrm{Hom}(f^*\underline{B},\underline{A}) \cong \mathrm{Hom}(\underline{B}, f_*\underline{A})$$

Special cases:

1. $\underline{A} = f^*\underline{B}$:

$$\mathrm{Hom}(f^*\underline{B}, f^*\underline{B}) \cong \mathrm{Hom}(\underline{B}, f_*f^*\underline{B})$$
$$\mathrm{id} \mapsto \underbrace{(\underline{B}\to f_*f^*\underline{B})}_{\text{canonical map}}$$

2. $\underline{B} = f_*\underline{A}$:

$$\mathrm{Hom}(f^*f_*\underline{A}, \underline{A}) \cong \mathrm{Hom}(f_*\underline{A}, f_*\underline{A})$$
$$\underbrace{(f^*f_*\underline{A}\to \underline{A})}_{\text{canonical map}} \leftarrow\!\shortmid \mathrm{id}$$

Summary:

1. $\underline{H}^i(\underline{IC}^{\bullet})_x = 0$, for $i > \bar{p}(k) - n, x\in U_{k+1}$

2. $\underline{H}^i(\underline{IC}^\bullet|_{U_{k+1}}) \to \underline{H}^i(i_{k*}i_k^* IC^\bullet|_{U_{k+1}})_x$ is an isomorphism, for $i \leqslant \bar{p}(k) - n$, $x \in X_{n-k} \setminus X_{n-k-1}$.

$$j_k \colon U_{k+1} \setminus U_k \hookrightarrow U_{k+1}$$

Definition 5.4. *A complex of sheaves $\underline{A}^\bullet$ is **cohomologically locally constant**, if $\underline{H}^i(\underline{A}^\bullet)|_{X_{n-k} \setminus X_{n-k-1}}$ are locally constant for all i, k.*
*$\underline{A}^\bullet$ is called **constructible**, if it is cohomologically locally constant and all stalks of $\underline{H}^i(\underline{A}^\bullet)$ are finitely generated.*

Fact: $\underline{IC}_{\bar{p}}(X)$ is constructable.
We define $D_c^b(X) \subset D(X)$ to be the subcategory of bounded constructable complex of sheaves.

6 Axiomatic characterization

Definition 6.1. *Let $\underline{A}^\bullet \in D_c^b(X)$. We say that $\underline{A}^\bullet$ satisfies [AX] in $D_c^b(X)$, iff:*

[AX0 *(Normalization)]* $\underline{A}^\bullet|_{U_2} \cong \underline{\mathbb{R}}_{U_2}[n]$

[AX1 *(Lower bound)]* $\underline{H}^i(\underline{A}^\bullet) = 0, \forall i < -n$

[AX2 *(Stalk vanishing)]* $\underline{H}^i(\underline{A}|_{U_{k+1}}) = 0, \forall i > \bar{p}(k) - n$

[AX3 *(Attaching axiom)]* $\underline{H}^i(j_k^* \underline{A}^\bullet|_{U_{k+1}}) \to \underline{H}^i(j_k^* R i_{k*} i_k^* \underline{A}^\bullet|_{U_{k+1}})$ *is an isomorphism for $i \leqslant \bar{p}(k) - n$, where Ri_{k*} is the derived functor of i_{k*}.*